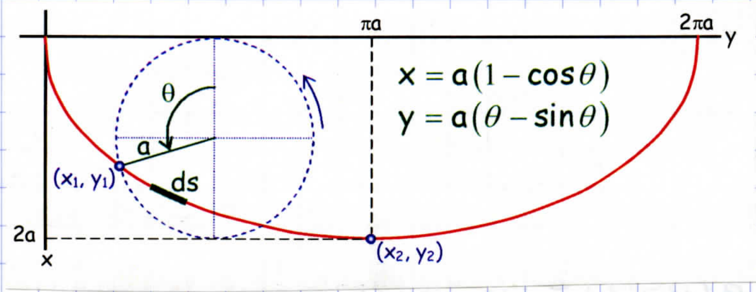
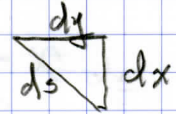
TMS 6.5

REEXAMINE THE BRACHISTOCHRONE AND SHOW THAT THE TIME REQUIRED FOR A PARTICLE TO MOVE TO THE MINIMUM POINT IS INDEPENDENT OF THE STARTING POINT AND IS

$$t_{\text{TO MIN}} = \pi\sqrt{\frac{a}{g}}$$



LET THE BEAD START AT A RANDOM x_1, y_1 AND WRITE THE DISTANCE ALONG THE CURVE



$$s = v t$$

WHERE $\quad ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$ds = \sqrt{1 + (y')^2}\, dx$$

THE VELOCITY IS THAT STARTING FROM REST AT X_1

$$v = \sqrt{2g(x - x_1)} \qquad (v_{x_1} = 0, \text{ BUT DON'T LOSE } x_1)$$

$\frac{1}{2}mv^2 = mgh$

$v = \sqrt{gh}$

$$\Rightarrow \quad dt = \frac{ds}{v} = \frac{\sqrt{1 + (y')^2}}{\sqrt{2g(x - x_1)}}\, dx$$

USE THE DERIVATIVES TO WRITE THE PATH ALONG THE CYCLOID:

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = a(1 - \cos\theta)\frac{d\theta}{dx}$$

AND

$$\frac{d}{dx}(x = a - a\cos\theta) \Rightarrow 1 = +a\sin\theta\,\frac{d\theta}{dx}$$

$$\Rightarrow \frac{d\theta}{dx} = \frac{1}{a\sin\theta}$$

GIVING

$$y' = \frac{1 - \cos\theta}{\sin\theta}$$

$$(y')^2 = \frac{(1 - \cos\theta)^2}{\sin^2\theta} = \frac{(1 - \cos\theta)\cancel{(1 - \cos\theta)}}{\cancel{(1 - \cos\theta)}(1 + \cos\theta)} \longrightarrow$$

NOTE BOOK

$$(y')^2 = \frac{1-\cos\theta}{1+\cos\theta}$$

AND

$$1+(y')^2 = 1+\frac{1-\cos\theta}{1+\cos\theta} = \frac{1+\cancel{\cos\theta}+1-\cancel{\cos\theta}}{1+\cos\theta} = \frac{2}{1+\cos\theta}$$

THE EXPRESSION WE NEED TO INTEGRATE IS

$$dt = \sqrt{\frac{\frac{2}{1+\cos\theta}}{2g(x-x_1)}}\,dx$$

SO WE NEED TO WRITE $x - x_1$ IN TERMS OF θ:

$$x - x_1 = (a - a\cos\theta) - (a - a\cos\theta_1)$$

$$x - x_1 = a(\cos\theta_1 - \cos\theta)$$

AND FROM EARLIER

$$dx = a\sin\theta\, d\theta$$

THUS

$$dt = \sqrt{\frac{1}{ga(\cos\theta_1-\cos\theta)(1+\cos\theta)}}\; a\sin\theta\, d\theta$$

$$dt = \sqrt{\frac{a}{g}}\sqrt{\frac{\sin^2\theta}{(\cos\theta_1-\cos\theta)(1+\cos\theta)}}\, d\theta$$

$$dt = \sqrt{\frac{a}{g}}\sqrt{\frac{(1-\cos\theta)\cancel{(1+\cos\theta)}}{(\cos\theta_1-\cos\theta)\cancel{(1+\cos\theta)}}}\, d\theta$$

$$dt = \sqrt{\frac{a}{g}}\sqrt{\frac{1-\cos\theta}{\cos\theta_1-\cos\theta}}\, d\theta$$

NOTE THAT

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \Rightarrow \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \Rightarrow 1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$$

Thus, substituting these into dt

$$dt = \sqrt{\frac{a}{g}} \sqrt{\frac{2\sin^2\left(\frac{\theta}{2}\right)}{\left[2\cos^2\left(\frac{\theta_1}{2}\right)-1\right]-\left[2\cos^2\left(\frac{\theta}{2}\right)-1\right]}}\, d\theta$$

$$dt = \sqrt{\frac{a}{g}} \frac{\pm \sin\left(\frac{\theta}{2}\right)}{\sqrt{\cos^2\left(\frac{\theta_1}{2}\right)-\cos^2\left(\frac{\theta}{2}\right)}}\, d\theta$$

Now substitute $z = \cos\left(\frac{\theta}{2}\right)$ and $dz = -\frac{1}{2}\sin\left(\frac{\theta}{2}\right)d\theta$

$d\theta = 2\sin\left(\frac{\theta}{2}\right)d\theta$

$$dt = \sqrt{\frac{a}{g}} \frac{\mp 2\, dz}{\sqrt{\cos^2\left(\frac{\theta_1}{2}\right) - z^2}}$$

Convert the limits

Starting point: $x_1 \to \theta_1$

Ending point $x_f = 2a = a(1-\cos\theta)$

$\Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$

Now integrate

$$\int_0^t dt = \sqrt{\frac{a}{g}} \int_{\theta_1}^{\pi} \frac{\pm dz}{\sqrt{\underbrace{\cos^2\left(\frac{\theta_1}{2}\right)}_{\text{"}a^2\text{"}} - z^2}}$$

Which looks like

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \text{ArcSin}\left(\frac{x}{a}\right) \qquad \text{Blue Book p 202, \# 115}$$

Giving

$$t_{\text{to min}} = \mp 2\sqrt{\frac{a}{g}} \left[\text{ArcSin}\left(\frac{z}{\cos\left(\frac{\theta_1}{2}\right)}\right)\right|_{\theta_1}^{\pi}$$

Replacing $z = \cos\left(\frac{\theta}{2}\right)$ gives

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TM5 Pr. 6.5 Continued

$$t_{\text{TO MIN}} = \mp 2\sqrt{\frac{a}{g}}\left\{ \text{ArcSin}\left[\frac{\cos\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta_1}{2}\right)}\right]\right\}\Bigg|_{\theta_1}^{\pi}$$

$$= \mp 2\sqrt{\frac{a}{g}}\left\{ \text{ArcSin}\left[\frac{\cancelto{0}{\cos\left(\frac{\pi}{2}\right)}}{\cos\left(\frac{\theta_1}{2}\right)}\right] - \text{ArcSin}\left[\cancelto{1}{\frac{\cos\left(\frac{\theta_1}{2}\right)}{\cos\left(\frac{\theta_1}{2}\right)}}\right]\right\}$$

$$= \mp 2\sqrt{\frac{a}{g}}\left[-\text{ArcSin}(1)\right]$$

$$= \mp 2\sqrt{\frac{a}{g}}\left[-\frac{\pi}{2}\right]$$

Since $t_{\text{TO MIN}}$ must be positive,

$$\boxed{t_{\text{TO MIN}} = \sqrt{\frac{a}{g}}\,\pi}$$ Whew!